

# FIXED POINT THEOREMS IN $G$ -METRIC SPACE.

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**ABSTRACT.** In this article, we present a new type of fixed point for single valued mapping in a  $G$ -complete  $G$ -metric space.

## 1. INTRODUCTION AND PRELIMINARIES

The importance of fixed point in mathematical analysis and topology is no longer to be established. For instance, it is used to determine existence and uniqueness of solutions of differential and integral equations. Initially stated in the metric space setting, fixed point theory has found its way in more general spaces, even though most of them are metric-like. One of these general spaces, space of interest for our study, is the  $G$ -metric space, where many fixed point theorems have already been established, see [2, 3, 5, 6, 7, 8]. Throughout the years, different authors proposed different types of formulations, all expressing different contractive-type conditions and most of these contractions are Picard operators and therefore lead to the uniqueness of the fixed point. However, for a given self mapping  $T$  on set  $X$ , if the set of fixed point of  $T$  is nonempty and the sequence of successive approximation for any initial point converges to a fixed point of  $T$ ,  $T$  is called a weakly Picard operator. We present here new fixed point results in a  $G$ -complete  $G$ -metric space, both in the Picard and weakly Picard cases. The mappings we consider satisfy a rational type almost contraction. The elementary facts about  $G$ -metric spaces can be found in [6]. We give here a shortened form of these prerequisites.

**Definition 1.1.** (Compare [6, Definition 3]) Let  $X$  be a nonempty set, and let the function  $G : X \times X \times X \rightarrow [0, \infty)$  satisfy the following properties:

- (G1)  $G(x, y, z) = 0$  if  $x = y = z$  whenever  $x, y, z \in X$ ;
- (G2)  $G(x, x, y) > 0$  whenever  $x, y \in X$  with  $x \neq y$ ;
- (G3)  $G(x, x, y) \leq G(x, y, z)$  whenever  $x, y, z \in X$  with  $z \neq y$ ;
- (G4)  $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ , (symmetry in all three variables);
- (G5)

$$G(x, y, z) \leq [G(x, a, a) + G(a, y, z)]$$

for any points  $x, y, z, a \in X$ .

Then  $(X, G)$  is called a  **$G$ -metric space**.

**Proposition 1.2.** (Compare [6, Proposition 6]) Let  $(X, G)$  be a  $G$ -metric space. Define on  $X$  the metric  $d_G$  by  $d_G(x, y) = G(x, y, y) + G(x, x, y)$  whenever  $x, y \in X$ . Then for a sequence  $(x_n) \subseteq X$ , the following are equivalent

- (i)  $(x_n)$  is  $G$ -convergent to  $x \in X$ .

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- (ii)  $\lim_{n,m \rightarrow \infty} G(x, x_n, x_m) = 0$ .
- (iii)  $\lim_{n \rightarrow \infty} d_G(x_n, x) = 0$ .
- (iv)  $\lim_{n \rightarrow \infty} G(x, x_n, x_n) = 0$ .
- (v)  $\lim_{n \rightarrow \infty} G(x_n, x, x) = 0$ .

**Proposition 1.3.** (Compare [6, Proposition 9])

In a  $G$ -metric space  $(X, G)$ , the following are equivalent

- (i) The sequence  $(x_n) \subseteq X$  is  $G$ -Cauchy.
- (ii) For each  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $G(x_n, x_m, x_m) < \varepsilon$  for all  $m, n \geq N$ .

**Definition 1.4.** (Compare [6, Definition 4]) A  $G$ -metric space  $(X, G, K)$  is said to be symmetric if

$$G(x, y, y) = G(x, x, y), \text{ for all } x, y \in X.$$

**Definition 1.5.** (Compare [6, Definition 9]) A  $G$ -metric space  $(X, G)$  is  $G$ -complete if every  $G$ -Cauchy sequence of elements of  $(X, G)$  is  $G$ -convergent in  $(X, G)$ .

We conclude this introductory part with:

**Definition 1.6.** A self mapping  $T$  defined on a  $G$ -metric space  $(X, G, K)$  is said to be **orbitally continuous** if and only if

$$\lim_{i \rightarrow \infty} T^{n_i} x = x^* \in X \implies Tx^* = \lim_{i \rightarrow \infty} TT^{n_i} x.$$

## 2. THE RESULTS

**Theorem 2.1.** Let  $(X, G)$  be a symmetric  $G$ -complete  $G$ -metric space and  $T$  be a mapping from  $X$  to itself. Suppose that  $T$  satisfies the following condition:

$$G(Tx, Ty, Tz) \leq \left( \frac{G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)}{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz) + 1} \right) G(x, y, z), \quad (2.1)$$

for all  $x, y, z \in X$ . Then

- (a)  $T$  has at least one fixed point  $\xi \in X$ ;
- (b) for any  $x \in X$ , the sequence  $\{T^n x\}$   $G$ -converges to a fixed point;
- (c) if  $\xi, y^* \in X$  are two distinct fixed points, then

$$G(\xi, y^*, y^*) = G(\xi, \xi, y^*) \geq \frac{1}{3}.$$

*Proof.* Let  $x_0 \in X$  be arbitrary and construct the sequence  $\{x_n\}$  such that  $x_{n+1} = Tx_n$ . We have, for the triplet  $(x_n, x_{n+1}, x_{n+1})$ , and by setting  $d_n = G(x_n, x_{n+1}, x_{n+1})$ , we have:

$$\begin{aligned} d_n &= G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \left( \frac{G(x_n, x_n, x_n) + 2G(x_{n-1}, x_{n+1}, x_{n+1})}{d_{n-1} + 2d_n + 1} \right) d_{n-1} \\ &\leq \left( \frac{2d_{n-1} + 2d_n}{d_{n-1} + 2d_n + 1} \right) d_{n-1}. \end{aligned}$$

If we set

$$\alpha_n = \frac{2d_{n-1} + 2d_n}{d_{n-1} + 2d_n + 1},$$

we get, iteratively

$$\begin{aligned} d_n &\leq \alpha_n d_{n-1} \\ &\leq \alpha_n \alpha_{n-1} d_{n-2} \\ &\vdots \\ &\leq \alpha_n \alpha_{n-1} \cdots \alpha_1 d_0. \end{aligned}$$

It is clear that the sequence  $\{\alpha_n\}$  is a non-increasing sequence of positive reals, so

$$\alpha_n \alpha_{n-1} \cdots \alpha_1 \leq \alpha_1^n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore

$$\lim_{n \rightarrow \infty} \alpha_n \alpha_{n-1} \cdots \alpha_1 = 0,$$

hence

$$\lim_{n \rightarrow \infty} d_n = 0.$$

For any  $m, n \in \mathbb{N}, m > n$ , since we have

$$G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} G(x_{n+i}, x_{n+i+1}, x_{n+i+1}),$$

which translate to

$$G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} d_{n+i},$$

we obtain

$$G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} [(\alpha_{n+i} \cdots \alpha_1) d_0].$$

Put  $b_k = \alpha_k \cdots \alpha_1$  and observe that

$$\lim_{k \rightarrow \infty} \frac{b_{k+1}}{b_k} = 0, \text{ i.e. the series } \sum_{k=0}^{\infty} b_k < \infty,$$

therefore

$$\sum_{i=0}^{m-n} (\alpha_{n+i} \cdots \alpha_1) \rightarrow 0 \text{ as } m \rightarrow \infty.$$

In other words,  $\{x_n\}$  is a  $G$ -Cauchy sequence so  $G$ -converges to some  $\xi \in X$ .

Claim:  $\xi$  is a fixed point of  $T$ .

For the triplet  $(x_{n+1}, T\xi, T\xi)$  in (2.1), we get

$$G(x_{n+1}, T\xi, T\xi) \leq \left( \frac{G(x_n, \xi, \xi) + 2G(x_n, T\xi, \xi)}{d_n + 2G(\xi, T\xi, T\xi) + 1} \right) G(x_n, T\xi, T\xi) \quad (2.2)$$

On taking the limit on both sides of (2.2), we have  $G(\xi, T\xi, T\xi) = 0$ , thus  $T\xi = \xi$ .

If  $\kappa$  is a fixed point of  $T$  with  $\kappa \neq \xi$ , then

$$\begin{aligned} G(\xi, \kappa, \kappa) &= G(T\xi, T\kappa, T\kappa) \\ &\leq [G(\xi, \kappa, \kappa) + 2G(\xi, \kappa, \kappa)]G(\xi, \kappa, \kappa) \\ &\leq 3[G(\xi, \kappa, \kappa)]^2. \end{aligned}$$

Therefore,

$$G(\xi, \kappa, \kappa) = G(\xi, \xi, \kappa) \geq \frac{1}{3}.$$

□

**Example 2.2.** Let  $X = \{0, 1/2, 1\}$  and let  $G : X^3 \rightarrow [0, \infty)$  be defined by

$$\begin{aligned} G(0, 1, 1) &= 6 = G(1, 0, 0), \quad G(0, 1/2, 1/2) = 4 = G(1/2, 0, 0) \\ G(1/2, 1, 1) &= 5 = G(1, 1/2, 1/2), \quad G(0, 1/2, 1) = 15/2 \\ G(x, x, x) &= 0 \quad \forall x \in X, \end{aligned}$$

and  $G$  is a symmetric function of its three variables.  $(X, G)$  is  $G$ -complete.

Let  $T : X \rightarrow X$  be defined by  $T0 = 0$ ,  $T1/2 = 1/2$ ,  $T1 = 0$ .

$$G(T0, T1/2, T1/2) = G(0, 1/2, 1/2) = 4; \quad G(T0, T1, T1) = G(0, 0, 0) = 0;$$

$$G(T1/2, T1, T1) = G(1/2, 0, 0) = 4; \quad G(T0, T1/2, T1) = G(0, 1/2, 0) = 4,$$

and we have

$$\begin{aligned} 4 &= G(T0, T1/2, T1/2) = G(0, 1/2, 1/2) \\ &\leq \frac{G(T0, 1/2, 1/2) + G(0, T1/2, 1/2) + G(0, 1/2, T1/2)}{G(0, T0, T0) + 2G(1/2, T1/2, T1/2) + 1} \\ &\quad \times G(0, 1/2, 1/2) \\ &= \frac{4 + 4 + 4}{1} 4 = 48. \end{aligned}$$

Again,

$$\begin{aligned} 0 &= G(T0, T1, T1) = G(0, 0, 0) \\ &\leq \frac{G(T0, 1, 1) + G(0, T1, 1) + G(0, 1, T1)}{G(0, T0, T0) + 2G(1, T1, T1) + 1} \times G(0, 1, 1) \\ &= \frac{6 + 6 + 6}{13} 6. \end{aligned}$$

Also,

$$\begin{aligned}
4 &= G(T1/2, T1, T1) = G(1/2, 0, 0) \\
&\leq \frac{G(T1/2, 1, 1) + G(1/2, T1, 1) + G(1/2, 1, T1)}{G(1/2, T1/2, T1/2) + 2G(1, T1, T1) + 1} \\
&\quad \times G(1/2, 1, 1) \\
&= \frac{6 + 15}{13} 5.
\end{aligned}$$

Finally,

$$\begin{aligned}
4 &= G(T0, T1/2, T1) = G(0, 1/2, 0) \\
&\leq \frac{G(T0, 1/2, 1) + G(0, T1/2, 1) + G(0, 1/2, T1)}{G(0, T0, T0) + G(1/2, T1/2, T1/2) + G(1, T1, T1) + 1} \\
&\quad \times G(0, 1/2, 1) \\
&= \frac{15 + 4}{7} \times \frac{15}{2}.
\end{aligned}$$

Therefore  $T$  satisfies all the conditions of Theorem 2.1. Also,  $T$  has two distinct fixed points  $\{0, 1/2\}$  and  $G(0, 1/2, 1/2) = G(1/2, 0, 0) = 4 \geq 1/3$ .

*Remark 2.3.* The map  $T$  defined in Theorem 2.1 belongs to the category of the so-called weakly Picard operator, as the uniqueness of the fixed is not guaranteed. Moreover, one could also just require  $G$  to be an arbitrary  $G$ -metric, i.e. not necessarily **symmetric**.

In the same style, we present the following result, in which the map  $T$  leaves exactly one point of  $X$  fixed. This is the Picard case.

**Theorem 2.4.** *Let  $(X, G)$  be a symmetric  $G$ -complete  $G$ -metric space and  $T$  be a mapping from  $X$  to itself. Suppose that  $T$  satisfies the following condition:*

$$G(Tx, Ty, Tz) \leq \alpha \left( \frac{\min\{G(y, Ty, Ty), G(z, Tz, Tz)\}[1 + G(x, Tx, Tx)]}{1 + G(x, y, z)} \right) + \beta G(x, y, z), \quad (2.3)$$

for all  $x, y, z \in X$  where  $\alpha, \beta$  are nonnegative reals, satisfying

$$\alpha + \beta < 1.$$

Then  $T$  leaves exactly one point of  $X$  fixed.

*Proof.* Let  $x_0 \in X$  be arbitrary and construct the sequence  $\{x_n\}$  such that  $x_{n+1} = Tx_n$ . We have, for the triplet  $(x_n, x_{n+1}, x_{n+1})$ , and by setting  $d_n = G(x_n, x_{n+1}, x_{n+1})$ , we have:

$$\begin{aligned}
d_n &= G(Tx_{n-1}, Tx_n, Tx_n) \\
&\leq \frac{\alpha d_n [1 + d_{n-1}]}{1 + d_{n-1}} + \beta d_{n-1},
\end{aligned}$$

i.e.

$$d_n \leq \frac{\beta}{1-\alpha} d_{n-1}. \quad (2.4)$$

By usual procedure from (2.4), since  $(\frac{\beta}{1-\alpha}) < 1$ , it follows that  $\{T^n x_0\}$  is a  $G$ -Cauchy sequence. By  $G$ -completeness of  $X$ , there exists  $x^* \in X$  such that  $T^n x_0$   $G$ -converges to  $x^*$ . The uniqueness of  $x^*$  is given for free by the condition (2.3).  $\square$

We present, without proof, the following genralisation of Theorem 2.4.

**Theorem 2.5.** *Let  $(X, G)$  be a symmetric  $G$ -complete  $G$ -metric space and  $T$  be a mapping from  $X$  to itself. Suppose that  $T$  satisfies the following condition:*

$$\begin{aligned} G(Tx, Ty, Tz) \leq & a_1 \left( \frac{G(y, Ty, Ty)[1 + G(x, Tx, Tx)]}{1 + G(x, y, z)} \right) \\ & + a_2 \left( \frac{G(z, Tz, Tz)[1 + G(x, Tx, Tx)]}{1 + G(x, y, z)} \right) + a_3 G(x, y, z) \end{aligned} \quad (2.5)$$

for all  $x, y, z \in X$  where  $a_i := a_i(x, y, z), i = 1, 2, 3$ , are nonnegative functions such that for arbitrary  $0 < \lambda_1 < 1$ :

$$a_1(x, y, z) + a_2(x, y, z) + a_3(x, y, z) = \sum_{i=1}^3 a_i(x, y, z) \leq \lambda_1.$$

Then  $T$  leaves exactly one point of  $X$  fixed.

Another genralisation of Theorem 2.4 is provided by the following:

**Theorem 2.6.** *Let  $(X, G)$  be a symmetric  $G$ -complete  $G$ -metric space where  $T$  is an an orbitally continuous mapping from  $X$  to itself. If it is the case that  $T$  satisfies the following condition:*

$$\begin{aligned} G(Tx, Ty, Tz) \leq & a_1 G(x, y, z) + a_2 [G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)] \\ & + a_3 [G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)] \\ & + a_4 \min\{G(y, Ty, Ty), G(z, Tz, Tz)\} \frac{[1 + G(x, Tx, Tx)]}{1 + G(x, y, z)} \\ & + a_5 G(Tx, y, z) [1 + G(x, Ty, z) + G(x, y, Tz)] [1 + G(x, y, z)]^{-1} \\ & + a_6 G(x, y, z) [1 + G(x, Tx, Tx) + G(Tx, y, z)] [1 + G(x, y, z)]^{-1} \\ & + a_7 G(Tx, y, z) \end{aligned} \quad (2.6)$$

for all  $x, y, z \in X$  where  $a_i := a_i(x, y, z), i = 1, \dots, 7$ , are nonnegative functions such that for arbitrary  $0 < \lambda_1 < 1$ :

$$a_1(x, y, z) + 3a_2(x, y, z) + 4a_3(x, y, z) + a_4(x, y, z) + a_6(x, y, z) \leq \lambda_1,$$

then  $T$  leaves at least one point of  $X$  fixed.

*Proof.* Let  $x_0 \in X$  be arbitrary and construct the sequence  $\{x_n\}$  such that  $x_{n+1} = Tx_n$ . We have, for the triplet  $(x_n, x_{n+1}, x_{n+1})$ , and by setting  $d_n = G(x_n, x_{n+1}, x_{n+1})$ , we have:

$$\begin{aligned} d_{n+1} &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq a_1 d_n + a_2 [d_n + 2d_{n+1}] + a_3 [2d_n + 2d_{n+1}] + a_4 d_{n+1} + a_6 d_n. \end{aligned}$$

i.e.

$$d_{n+1} \leq \frac{a_1 + a_2 + 2a_3 + a_6}{1 - (2a_2 + 2a_3 + a_4)} d_n. \quad (2.7)$$

By usual procedure from (2.7), since

$$a_1(x, y, z) + 3a_2(x, y, z) + 4a_3(x, y, z) + a_4(x, y, z) + a_6(x, y, z) < 1,$$

it follows that  $\{T^n x_0\}$  is a  $G$ -Cauchy sequence. By  $G$ -completeness of  $X$ , there exists  $x^* \in X$  such that  $T^n x_0$   $G$ -converges to  $x^*$ . Since  $X$  is  $G$ -complete. Obviously  $x^*$  is the desired fixed point by orbitally continuity of  $T$ . □

**Example 2.7.** Let  $[a, b]$  where  $1 < a < b$ . Let the  $G$ -metric  $G$  be given on  $[a, b]$  as:

$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}.$$

Let  $T : X \rightarrow X$  be defined as follows:

$$Tx = x + \frac{1}{x} - \frac{1}{b}.$$

Let  $1 - \frac{1}{b^2} \leq \alpha < 1$  and  $\beta, \gamma, \delta, \lambda, \mu$  and  $L$  be arbitrary nonnegative reals such that

$$a_1(x, y, z) \leq \alpha, \quad a_2(x, y, z) \leq \beta, \quad a_3(x, y, z) \leq \gamma, \quad a_4(x, y, z) \leq \delta,$$

and

$$a_5(x, y, z) \leq \lambda, \quad a_6(x, y, z) \leq \mu, \quad a_7(x, y, z) \leq L$$

and

$$\alpha + 3\beta + 4\gamma + \mu + \delta < 1.$$

Here all the conditions of Theorem 2.6 are satisfied and it is readily seen that  $b$  is a fixed point of  $T$ .

In the extension of metric fixed point theory, generalization of metric spaces via complex valued ordered metric space, were introduced. The author plans to study more thoroughly, and with examples, fixed point results in the setting of ordered  $G$ -metric space in another paper [4] but we present here a first result of the kind.

Recall that we can define a partial order  $\preceq$  on the set  $\mathbb{C}$  of complex numbers by setting, for any  $z_1, z_2 \in \mathbb{C}$ ,

$$z_1 \preceq z_2 \iff \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2).$$

Moreover, on partial ordered  $G$ -metric space, the convergence of a sequence is interpreted in the canonical way, i.e. for a sequence  $\{x_n\} \subseteq (X, G, \preceq)$  where  $(X, G, \preceq)$  is a partial ordered complex valued  $G$ -metric space,

$$x_n \text{ } G\text{-converges to } x^* \iff \forall c \in \mathbb{C}, \text{ with } 0 \preceq c, \exists n_0 \in \mathbb{N} : \forall n > n_0 \quad G(x^*, x_n, x_n) \preceq c.$$

Similarly for  $G$ -Cauchy sequences. Furthermore, a self mapping  $T$  defined on a partial ordered  $G$ -metric space  $(X, G, \preceq)$  is nondecreasing if  $Tx \preceq Ty$  whenever  $x \preceq y$ , for  $x, y \in X$ . We then state the result:

**Theorem 2.8.** *Let  $(X, G, \preceq)$  be a symmetric,  $G$ -complete, complex valued  $G$ -metric space. Assume that if  $\{x_n\}$  is a nondecreasing sequence of elements of  $X$  such that  $x_n$   $G$ -converges to  $x^*$ , then  $x_n \preceq x^*$  for all  $n \in \mathbb{N}$ . Let  $T : X \rightarrow X$  be a nondecreasing mapping such that:*

$$\begin{aligned} G(Tx, Ty, Tz) \preceq & a_1 G(x, y, z) + a_2 [G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)] \\ & + a_3 [G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)] \\ & + a_4 \min\{G(y, Ty, Ty), G(z, Tz, Tz)\} \frac{[1 + G(x, Tx, Tx)]}{1 + G(x, y, z)} \\ & + a_5 G(Tx, y, z) [1 + G(x, Ty, z) + G(x, y, Tz)] [1 + G(x, y, z)]^{-1} \\ & + a_6 G(x, y, z) [1 + G(x, Tx, Tx) + G(Tx, y, z)] [1 + G(x, y, z)]^{-1} \\ & + a_7 G(Tx, y, z) \end{aligned} \quad (2.8)$$

for all  $x \preceq y \preceq z \in X$  where  $a_i := a_i(x, y, z)$ ,  $i = 1, \dots, 7$ , are nonnegative functions such that for arbitrary  $0 < \lambda_1 < 1$ :

$$a_1(x, y, z) + 3a_2(x, y, z) + 4a_3(x, y, z) + a_4(x, y, z) + a_6(x, y, z) \leq \lambda_1.$$

If there exists  $x_0 \in X$  with  $x_0 \preceq Tx_0$ , then  $T$  leaves at least one point of  $X$  fixed.

*Proof.* It is very easy to see that the sequence of iterates  $T^n x_0$ ,  $n = 1, 2, \dots$ , is nondecreasing and  $G$ -converges to some  $x^* \in X$ . Therefore  $x_n \preceq x^*$  for all  $n \in \mathbb{N}$ . Now applying (2.8) to the triplet  $(x_{n+1}, Tx^*, Tx^*)$  and taking the limit as  $n \rightarrow \infty$ , we have:

$$\begin{aligned} G(x^*, Tx^*, Tx^*) &= G(Tx_n, Tx^*, Tx^*) \\ &\preceq 2a_2 G(x^*, Tx^*, Tx^*) + 2a_3 G(x^*, Tx^*, Tx^*) + a_4 G(x^*, Tx^*, Tx^*) \\ &= [2a_2 + 2a_3 + a_4] G(x^*, Tx^*, Tx^*). \end{aligned}$$

Since  $2a_2 + 2a_3 + a_4 < 1$ , this is a contradiction unless  $G(x^*, Tx^*, Tx^*) = 0$ , and hence  $Tx^* = x^*$ . □

**Example 2.9.** Let  $X = [1.5, 2]$  with the usual partial order “ $\leq$ ”. Let the  $G$ -metric  $G$  be given on  $X$  as:

$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}(1 + i).$$

Let  $T : X \rightarrow X$  be defined as follows:

$$Tx = \begin{cases} 1.81, & \text{if } 1.5 \leq x < 1.75, \\ x + \frac{1}{x} - \frac{1}{2}, & \text{if } 1.75 \leq x \leq 2. \end{cases}$$

Let  $\alpha \in [\frac{3}{4}, 1)$ ,  $\beta = \gamma = \delta = \mu = \lambda = 0$  and  $L \geq 3$  be arbitrary nonnegative reals such that

$$a_1(x, y, z) \leq \alpha, \quad 0 = a_2(x, y, z) \leq \beta, \quad 0 = a_3(x, y, z) \leq \gamma, \quad 0 = a_4(x, y, z) \leq \delta,$$



and

$$0 = a_5(x, y, z) \leq \lambda, \quad a_6(x, y, z) \leq \mu, \quad a_7(x, y, z) \leq L.$$

Here all the conditions of Theorem 2.8 are satisfied and it is readily seen that 2 is a fixed point of  $T$ .

*Remark 2.10.* In the abover theorem, one could observe that there is no need of imposing any type of continuity on the map  $T$ . It is also good to mention at this point that complex valued  $G$ -metric space have close similarities with cone  $G$ -cone metric spaces( see[1]) even though both spaces are very different. Moreover the rational contraction we considered is better applicable and understood when studied in the complex valued case.

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